Angular Momentum and Discrete Symmetries in the Spinor Bethe-Salpeter Equation

W. Bauhoff

Institut für Theoretische Physik, Universität Tübingen

Z. Naturforsch. **30 a**, 395–401 [1975]; received January 8, 1975)

The group-theoretical basis for the angular momentum reduction of the spinor Bethe-Salpeter equation is discussed. We construct amplitude of definite angular momentum by analogy to the nonrelativistic problem and discuss the differences. The additional decomposition because of discrete symmetries is explained. Finally, we relate the states to those obtained in the nonrelativistic quark model.

I. Introduction and Fundamentals

It is well known that the Bethe-Salpeter (BS) equation describing bound states is invariant under the little group of the total momentum p_μ of the bound state, that is under three-dimensional rotations for time-like p_μ. Hence the solutions can be classified according to their angular momentum j and the equation may be decomposed into a set of equations corresponding to definite values of the angular momentum operators $\boldsymbol{J}^2$ and J_3. In the scalar case this is achieved simply by expanding the wave-functions in terms of spherical harmonics. In the spinor case which we will consider in the present paper, the problem is much more intricate since we must combine the space dependent part of the amplitudes with their Dirac structure. Besides the angular momentum, we have as additional symmetries the parity $\mathcal{P}$, and the charge conjugation $\mathcal{C}$ in the case of the equal mass fermion-antifermion problem. So the set of coupled equations obtained by the angular momentum decomposition is split up into four distinct ones, according to the eigenvalues of the states with respect to $\mathcal{P}$ and $\mathcal{C}$.

Recently, some additional symmetries of the spinor BS-equations have been discussed, for example the reflection of the total momentum $p_\mu \to -p_\mu$ [1] but we will not discuss it here.

The importance of the spinor BS-equation must not be stressed here. It will be sufficient to say that in all bound-state models of elementary particles such as the quark model [2] and the nonlinear spinor theory [3], the fundamental fields are assumed to be spinors. So it is not astonishing that the kinematical analysis of the spinor BS-equation has been considered by several authors [4–6]. But nevertheless there remain some problems to be discussed, for example the meaning of the degeneracy parameters in the angular momentum decomposition or the relation to the state assignments of the nonrelativistic quark model. It is the purpose of this paper to clarify these points and to give a useful review.

The fermion-antifermion BS-amplitude is defined by

$$\tau(x_1, x_2) = \langle 0 \,|\, T\, \psi(x_1)\, \overline{\psi}(x_2) \,|\, a \rangle\,. \tag{1.1}$$

For a bound state $|a\rangle$ the amplitude satisfies the homogeneous BS-equation which reads in coordinate space

$$(-i\gamma^\mu\, \partial/\partial x_1{}^\mu + m)\, \tau(x_1, x_2)\, (-i\gamma^\nu\, \overleftarrow{\partial}/\partial x_2{}^\nu + m) = \int \mathfrak{K}(x_1, x_2; x_3, x_4)\, \tau(x_3, x_4)\, \mathrm{d}x_3\, \mathrm{d}x_4\,. \tag{1.2}$$

$\mathfrak{K}(x_1, x_2; x_3, x_4)$ is the interaction kernel which is assumed to be invariant under Lorentz transformations including reflections. It should be noted that we do not restrict the discussion to ladder-like interactions where

$$\mathfrak{K}(x_1, x_2; x_3, x_4) \sim \mathfrak{K}'(x_1 - x_2)\, \delta(x_1 - x_3)\, \delta(x_2 - x_4)$$

The field operator $\psi(x)$ transforms under Lorentz transformations $x' = \Lambda x$ according to

$$U(\Lambda)\, \psi(x)\, U^{-1}(\Lambda) = S^{-1}(\Lambda)\, \psi(\Lambda x) \tag{1.3}$$

where $S(\Lambda)$ is the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation matrix of the homogeneous Lorentz group. Hence we have for an infinitesimal Lorentz rotation

$$x'^\mu = x^\mu + a_\nu{}^\mu\, x^\nu: \tag{1.4}$$
$$S(\Lambda) = 1 - (i/2)\, a^{\mu\nu}\, \Sigma_{\mu\nu}\,, \quad \Sigma_{\mu\nu} = (i/4)\, [\gamma_\mu, \gamma_\nu]\,.$$

So we get from (1.4) the commutation relation of $\psi(x)$ with the generator $M_{\mu\nu}$ defined by

$$U(\Lambda) = 1 + (i/2)\, a^{\mu\nu}\, M_{\mu\nu}\,, \tag{1.5}$$

$$[M_{\mu\nu}, \psi(x)] = (i\, x_\mu\, \partial_\nu - i\, x_\nu\, \partial_\mu + \Sigma_{\mu\nu})\, \psi(x)\,. \tag{1.6}$$

Similarly, we have the commutation relation for the generator P_μ of translations $x'^\mu = x^\mu + a^\mu$:

$$[P_\mu, \psi(x)] = i\,\partial_\mu\,\psi(x)\,. \tag{1.7}$$

The corresponding formulas for $\bar{\psi}(x)$ are obtained trivially from (1.6), (1.7).

Next we want to construct the operators $\mathcal{P}_\mu$ and $\mathcal{M}_{\mu\nu}$ acting on $\tau(x_1, x_2)$ which are equivalent to P_μ and $M_{\mu\nu}$ acting on the state $|a\rangle$. That means we require:

$$\mathcal{P}_\mu\,\tau(x_1, x_2) = \langle 0\,|\,T\,\{\psi(x_1)\,\bar{\psi}(x_2)\}\,P_\mu\,|\,a\rangle\,, \tag{1,8}$$

$$\mathcal{M}_{\mu\nu}\,\tau(x_1, x_2) = \langle 0\,|\,T\,\{\psi(x_1)\,\bar{\psi}(x_2)\}\,M_{\mu\nu}\,|\,a\rangle\,. \tag{1.9}$$

From the commutation relations (1.6), (1.7) we find:

$$\mathcal{P}_\mu\,\tau(x_1, x_2) = -\frac{i}{2}\left(\frac{\partial}{\partial x_1{}^\mu} + \frac{\partial}{\partial x_2{}^\mu}\right)\tau(x_1, x_2)\,, \tag{1.10}$$

$$\mathcal{M}_{\mu\nu}\,\tau(x_1, x_2) = -i\left(x_{1\mu}\frac{\partial}{\partial x_1{}^\nu} - x_{1\nu}\frac{\partial}{\partial x_1{}^\mu} + x_{2\mu}\frac{\partial}{\partial x_2{}^\nu} - x_{2\nu}\frac{\partial}{\partial x_2{}^\mu}\right)\tau(x_1, x_2) + [\Sigma_{\mu\nu}, \tau(x_1, x_2)]\,. \tag{1.11}$$

Now we require $|a\rangle$ to be a state of definite momentum p_μ and definite angular momentum j. So we have

$$P_\mu\,|\,a\rangle = p_\mu\,|\,a\rangle\,,\quad P^\mu P_\mu\,|\,a\rangle = m^2\,|\,a_\mu\,, \tag{1.12}$$

$$J^\mu J_\mu\,|\,a\rangle = j(j+1)\,|\,a\rangle \tag{1.13}$$

with the Pauli-Lubanski spin vector

$$J_\mu = (1/2\,m)\,\varepsilon_{\mu\nu\varrho\sigma}\,P^\nu\,M^{\varrho\sigma} \tag{1.14}$$

(we are dealing with massive particles only). Translating (1.12) into an equation for $\tau(x_1, x_2)$ by means of (1.10) we have as solutions:

$$\tau(x_1, x_2) = \varphi(z)\,\exp\{-i\,p\cdot(x_1 + x_2)/2\} \tag{1.15}$$

with the relative coordinate $z = x_1 - x_2$.

The transcription of (1.13) into an equation for $\varphi(z)$ and its solution is greatly simplified if we pass to the rest frame $p_\mu = (m, 0, 0, 0)$. Then we have

$$J_0 = 0\,,\quad J_i = \tfrac{1}{2}\,\varepsilon_{ikl}\,M_{kl}\,. \tag{1.16}$$

Furthermore we define

$$L_i = \tfrac{1}{2}\,\varepsilon_{ikl}\,z_k\,\partial/\partial z_l\,, \tag{1.17}$$

$$\sigma_i' = \tfrac{1}{2}\,\varepsilon_{ikl}\,\Sigma_{kl} \tag{1.18}$$

and the condition for definite angular momentum reads:

$$\sum_{i=1}^{3}(L_i + \sigma_i' \otimes 1 - 1 \otimes \sigma_i'^T)^2\,\varphi(z) = j(j+1)\,\varphi(z)\,. \tag{1.19}$$

(1.19) may look somewhat strange, but it should be noted that

$$[\sigma_i', \varphi(z)] = (\sigma_i' \otimes 1 - 1 \otimes \sigma_i'^T)\,\varphi(z)\,. \tag{1.20}$$

Additionally we have a definite spin projection j_3:

$$L_3\,\varphi(z) + [\sigma_3', \varphi(z)] = j_3\,\varphi(z)\,. \tag{1.21}$$

The solution of (1.19), (1.21) will be discussed in Section 2.

In a similar manner we may discuss the discrete symmetries. Consider for example the space inversion $\bar{x} = i_S\,x = (x_0, -\boldsymbol{x})$. The transformation law of the field operator is:

$$U(i_S)\,\psi(x)\,U^{-1}(i_S) = \gamma_0\,\psi(\bar{x}) \tag{1.22}$$

where we have arbitrarily chosen a phase factor $\eta_P' = +1$. The corresponding operator $\mathcal{P}$ acting on the amplitude $\tau(x_1, x_2)$ is found from (1.22) to be [7]:

$$\mathcal{P}\,\tau(x_1, x_2) = \gamma_0\,\tau(\bar{x}_1, \bar{x}_2)\,\gamma_0\,. \tag{1.23}$$

If the bound state $|a\rangle$ has a definite parity η_P we get from (1.23) an eigenvalue equation which we will write down immediately for the relative coordinate wave function $\varphi(z)$:

$$\mathcal{P}\,\varphi(z) = \gamma_0\,\varphi(\bar{z})\,\gamma_0 = \eta_P\,\varphi(z)\,. \tag{1.24}$$

(1.24) is again valid only in the rest frame. We will deal with it in Section 3. Finally we consider the charge conjugation. The transformation law of the field operator is:

$$U(i_C)\,\psi(x)\,U^{-1}(i_C) = C\,\bar{\psi}^{\mathrm{T}}(x) \tag{1.25}$$

(C is the well-known charge conjugation matrix), where we have again chosen a phase factor $\eta_C' = +1$. In the by now familiar manner we deduce from (1.25) the operator $\mathscr{C}$ acting on the amplitude:

$$\mathscr{C}\,\tau(x_1, x_2) = C\,\tau^{\mathrm{T}}(x_2, x_1)\,C^{-1} \tag{1.26}$$

and the eigenvalue equation for a state with definite charge conjugation parity reads in the rest frame:

$$\mathscr{C}\varphi(z) = C\varphi^{\mathrm{T}}(-z)C^{-1} = \eta_C\,\varphi(z)\,. \qquad (1.27)$$

It will be more convenient, however, to investigate the combined transformation $\mathscr{C}\,\mathcal{P}$. This is also a symmetry in the Weyl case and so it is of some interest in nonlinear spinor theory. The eigenvalue equation is easily deduced from (1.24), (1.27)

$$\mathscr{C}\mathcal{P}\,\varphi(z) = C\,\gamma_0\,\varphi^{\mathrm{T}}(-z_0,\boldsymbol{z})\gamma_0\,C^{-1} = \eta_P\,\eta_C\,\varphi(z)\,. \qquad (1.28)$$

We will deal with (1.28) in Section 4.

2. Angular Momentum

We will now start with the construction of amplitudes with definite angular momentum and its projection, i. e. with the solution of (1.19), (1.20). In doing so we will be guided by the corresponding nonrelativistic problem of a bound state of two spin 1/2-particles: There one couples first the two individual spins, and then the resulting spin s is coupled to the orbital angular momentum l.

So we begin by looking for the solutions of

$$S^2\,\varphi(z) := \sum_{i=1}^{3}(\sigma_i' \otimes 1 - 1 \otimes \sigma_i'^{\mathrm{T}})^2\,\varphi(z) = s(s+1)\,\varphi(z)\,, \qquad (2.1)$$

$$S_3\,\varphi(z) := (\sigma_3' \otimes 1 - 1 \otimes \sigma_3'^{\mathrm{T}})\,\varphi(z) = s_3\,\varphi(z)\,. \qquad (2.2)$$

The amplitude $\varphi(z)$ may be expanded in terms of the sixteen elements Γ_i of the Dirac algebra which is isomorphic to the direct product of two Pauli algebras:

$$\Gamma \cong \Sigma \otimes P\,, \quad \begin{array}{l}\Sigma = \{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}\,,\\ P = \{\varrho_0,\varrho_1,\varrho_2,\varrho_3\}\,,\end{array} \quad \sigma_i = \varrho_i\,. \qquad (2.3)$$

Because of

$$\sigma_i' = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix} = \sigma_i \otimes \varrho_0\,. \qquad (2.4)$$

the Eqs. (2.1), (2.2) are diagonal in P. So it will be sufficient to expand $\varphi(z)$ with respect to Σ. This had to be expected since the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are the same for pure rotations.

So this section applies directly to the Weyl case which was discussed in [8]. Only for inversions the use of Dirac spinors is necessary.

Doing the algebra and dropping the primes, (2.1) is

$$\tfrac{1}{2}(3 - \sum_{i=1}^{3}\sigma_i \otimes \sigma_i^{\mathrm{T}})\,\varphi(z) = s(s+1)\varphi(z) \qquad (2.5)$$

where $\varphi(z)$ must now be expanded in the elements of Σ. The eigenstates are easily found to be

$$\begin{array}{ll}\sigma_0 = 1 & \text{for } s=0\,,\\ \sigma_1,\sigma_2,\sigma_2 & \text{for } s=1\,.\end{array} \qquad (2.6)$$

σ_0 is also a solution of (2.2). For $s=1$ we have to pass to the spherical basis [9]. This is achieved by defining

$$\sigma_{1\,s_3} = U_{s_3 i}\,\sigma_i \qquad (2.7)$$

with the unitary transformation matrix

$$U_{s_3 i} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix} \qquad (2.8)$$

σ_{1s_3} has now the definite spin projection s_3. To simplify the formulas we will define $\sigma_{00} := \sigma_0$.

So we have the expansion of $\varphi(z)$ in terms of eigenfunctions of S^2 and S_3:

$$\varphi(z) = \sum_{ss_3}\sigma_{ss_3}\,\varphi^{ss_3}(z)\,. \qquad (2.9)$$

Next we turn to the "orbital" part, i. e. we are looking for solutions of

$$L^2\,\varphi(z) = l(l+1)\,\varphi(z)\,, \qquad (2.10)$$

$$L_3\,\varphi(z) = l_3\,\varphi(z)\,. \qquad (2.11)$$

The solutions are the spherical harmonics Y_{ll_3} multiplied by a function not depending on the angles. Combining this with (2.9) we have

$$\varphi(z) = \sum_{\substack{ss_3\\ ll_3}}\sigma_{ss_3}\,Y_{ll_3}(\Omega_z)\,\varphi^{ss_3,ll_3}(z_0,|\boldsymbol{z}|)\,. \qquad (2.12)$$

For φ^{ss_3,ll_3} the Clebsch-Gordan expansion holds:

$$\varphi^{ss_3,ll_3}(z_0,|\boldsymbol{z}|) = \sum_{jj_3}\langle s\,s_3\,l\,l_3\,|\,j\,j_3\rangle\,\varphi^{sl}_{jj_3}(z_0,|\boldsymbol{z}|)\,. \qquad (2.13)$$

For the notation of the Clebsch-Gordan coefficients see [9]. Inserting (2.13) in (2.12) we can now immediately solve (1.19), (1.20): The amplitudes with definite angular momentum and spin projection are:

$$\varphi_{jj_3}(z) = \sum_{\substack{ss_3\\ ll_3}}\langle s\,s_3\,l\,l_3\,|\,j\,j_3\rangle\,\sigma_{ss_3}\,Y_{ll_3}(\Omega_z)\,\varphi^{sl}_{jj_3}(z_0,|\boldsymbol{z}|)\,. \qquad (2.14)$$

The inversion of (2.14) is easily obtained with the help of the orthogonality relations for the Clebsch-Gordan coefficients and the spherical harmonics with $\bar{\sigma}_{ss_3} = (-1)^s \sigma_{ss_3}$:

$$\varphi^{sl}_{jj_3}(z_0, |\boldsymbol{z}|) = \sum_{s_3 l_3} \langle s\, s_3\, l\, l_3 | j\, j_3 \rangle \int \mathrm{d}\Omega_z\, Y^*_{ll_3}(\Omega_z) \mathrm{Tr}[\bar{\sigma}_{ss_3} \varphi_{jj_3}(z)] \,. \tag{2.15}$$

Alternatively, the construction of the amplitude $\varphi_{jj_3}(z)$ can be done with help of the vector spherical harmonics $Y_{(jlj_3)i}(\Omega_z)$. This method avoids the introduction of the spherical basis and is of some computational advantage. The vector spherical harmonics are defined by [9]:

$$\begin{aligned} Y_{(jlj_3)1} &= -\frac{1}{\sqrt{2}} \langle 1 1\, l\, j_3 - 1 | j\, j_3 \rangle\, Y_{lj_3-1} + \frac{1}{\sqrt{2}} \langle 1 -1\, l\, j_3 + 1 | j\, j_3 \rangle\, Y_{lj_3+1} \\ Y_{(jlj_3)2} &= -\frac{i}{\sqrt{2}} \langle 1 1\, l\, j_3 - 1 | j\, j_3 \rangle\, Y_{lj_3-1} - \frac{i}{\sqrt{2}} \langle 1 -1\, l\, j_3 + 1 | j\, j_3 \rangle\, Y_{lj_3+1} \\ Y_{(jlj_3)3} &= \langle 1 0\, l\, j_3 | j\, j_3 \rangle\, Y_{lj_3} \,. \end{aligned} \tag{2.16}$$

These three equations can be summarized with help of the transformation matrix U:

$$Y_{(jlj_3)i} = \sum_{s_3} U^+_{is_3} \langle 1\, s_3\, l\, j_3 - s_3 | j\, j_3 \rangle\, Y_{lj_3 - s_3} \,. \tag{2.17}$$

Inverting (2.7) we get another expression for the amplitude $\varphi_{jj_3}(z)$:

$$\varphi_{jj_3} = Y_{jj_3} \varphi^{0j}_{jj_3} + \sum_{i=1}^{3} \sigma_i [Y_{(jj-1j_3)i}\, \varphi^{1j-1}_{jj_3} + Y_{(jjj_3)i}\, \varphi^{1j}_{jj_3} + Y_{(jj+1j_3)i}\, \varphi^{1j+1}_{jj_3}] \tag{2.18}$$

where we have omitted the arguments of the functions. This form is used in the applications to the spinor BS-equation.

The meaning of the parameters s and l is somewhat intricate. In the nonrelativistic problem they are the (algebraic) spin and the orbital angular momentum of the two particle bound state. In the relativistic problem this is no longer true. The notion of "orbital angular momentum" requires the interpretation of the state $|a\rangle$ as consisting of two interacting particles. But the field operators $\psi(x)$ has a simple particle interpretation only when it fulfills the free Dirac equation because it transforms under a nonunitary representation (1.3). So s and l have simply the meaning of degeneracy parameters. Furthermore they are not constants of the motion.

Since s and l do not have an intuitive meaning anymore it will be desired to give a more abstract derivation of (2.10). This is achieved by noting that the Pauli matrices are essentially Clebsch-Gordan coefficients [10]:

$$(\sigma_{ss_3})_{\alpha\beta} = \sqrt{2s+1}\, \langle \tfrac{1}{2} \beta\, s\, s_3 | \tfrac{1}{2} \alpha \rangle \tag{2.19}$$

where the matrix indices run through $-\frac{1}{2}$, $+\frac{1}{2}$.

Inserting this in (2.10) and changing the normalization a little, we get

$$\varphi_{jj_3}(z) = \sum_{\substack{ss_3 \\ ll_3}} \langle \tfrac{1}{2} \alpha\, s\, s_3 | \tfrac{1}{2} \beta \rangle \langle s\, s_3\, l\, l_3 | j\, j_3 \rangle\; Y_{ll_3}(\Omega_z)\, \varphi^{sl}_{jj_3}(z_0, |\boldsymbol{z}|) \tag{2.20}$$

and this is easily recognized as a form of the Wigner-Eckart theorem. $\varphi^{sl}_{jj_3}$ is identified as the reduced matrix element. In order to get the decomposition (2.20) we have no longer to refer to nonrelativistic concepts.

The reduced amplitudes $\varphi^{sl}_{jj_3}$ are still matrices in the algebra P. The complete reduction is now carried out by expanding them in terms of the ϱ_i. The connection with the usual Dirac algebra is found from the following formulas

$$1 = 1 \otimes 1\,, \quad \gamma_5 = 1 \otimes \varrho_1\,, \quad \gamma_0 = 1 \otimes \varrho_3\,, \quad \gamma_0 \gamma_5 = 1 \otimes \varrho_3 \varrho_1\,, \tag{2.21}$$

$$\varepsilon_{ijk} \sigma_{jk} = \sigma_i \otimes 1\,, \quad \sigma_{0i} = \sigma_i \otimes \varrho_1\,, \quad \gamma_i \gamma_5 = \sigma_i \otimes \varrho_3\,, \quad \gamma_i = \sigma_i \otimes \varrho_3 \varrho_1 \,. \tag{2.22}$$

Summarizing, the amplitude $\varphi(z)$ is decomposed with respect to the Dirac algebra

$$\varphi(z) = S(z) + \gamma_5 P(z) + \gamma_0 V_4(z) + \gamma_0 \gamma_5 A_4(z) + \sum_{i=1}^{3} \{\gamma_i V_i(z) + \gamma_i \gamma_5 A_i(z) + \sigma_{0i} U_i(z) + \varepsilon_{ijk} \sigma_{jk} T_i(z)\} . \tag{2.23}$$

Then the "scalar" amplitudes [i.e. those with a matrix structure listed in (2.21)] are proportional to a spherical harmonic, for example (putting $|\boldsymbol{z}| = r$)

$$S(z) = Y_{jj_3}(\Omega_z)\, s_{jj_3}(z_0, r) . \tag{2.24}$$

For the "vector" amplitudes we have an expansion in vector spherical harmonics which we get from (2.18) with a slight change of the notation, for example:

$$V_i(z) = Y_{(jj-1j_3)i}(\Omega_z)\, v_{jj_3}^{(-)}(z_0, r) + Y_{(jjj_3)i}(\Omega_z)\, v_{jj_3}^{(0)}(z_0, r) + Y_{(jj+1j_3)i}(\Omega_z)\, v_{jj_3}^{(+)}(z_0, r) . \tag{2.25}$$

Substituting this into the BS-equation (1.2) we get a set of equations for the amplitudes $s_{jj_3}, \ldots, v_{jj_3}^{(-)}, \ldots$ From now on we will omit the subscript $j j_3$.

Table 1. The different states of the system and the amplitudes belonging to them.

Parity $(-1)^j$						Parity $(-1)^{j+1}$					
Triplet $j=l-1$			Triplet $j=l+1$			Singlet $j=l$			Triplet $j=l$		
States	Particles	$j^{\eta_P \eta_C}$	States	Particles	$j^{\eta_P \eta_C}$	States	Particles	$j^{\eta_P \eta_C}$	States	Particles	$j^{\eta_P \eta_C}$
$^3P_0$	ε, π_N	0^{++}				$^1S_0$	η, π	0^{-+}			
$^3D_1$	ω, ϱ	1^{--}	$^3S_1$	ω, ϱ	1^{--}	$^1P_1$	?, B	1^{+-}	$^3P_1$	?, A 1	1^{++}
$^3F_2$	f, A 2	2^{++}	$^3P_2$	f, A 2	2^{++}	$^1D_2$		2^{-+}	$^3D_2$		2^{--}
$^3G_3$		3^{--}	$^3D_3$		3^{--}	$^1F_3$		3^{+-}	$^3F_3$		3^{++}
Amplitudes s_e, $v_0^{(4)}$, $a_e^{(0)}$, $t_0^{(0)}$, $v_e^{(+)}$, $v_e^{(-)}$, $u_e^{(+)}$, $u_c^{(-)}$						Amplitudes p_e, $a_e^{(4)}$, $v_0^{(0)}$, $u_0^{(0)}$, $a_0^{(+)}$, $a_0^{(-)}$, $t_e^{(+)}$, $t_e^{(-)}$			Amplitudes p_0, $a_0^{(4)}$, $v_e^{(0)}$, $u_e^{(0)}$, $a_e^{(+)}$, $a_e^{(-)}$, $t_0^{(+)}$, $t_0^{(-)}$		

The two triplets $j=l\pm 1$ are degenerate. We have included the corresponding isoscalar and isovector particles, taken from [16].

3. Parity

The set of sixteen coupled equations obtained after angular momentum reduction is decoupled into smaller sets because of the discrete symmetries. As the first one we investigate parity. We have to look for solutions of (1.24) and consider as a first step the algebraic part. Since γ_0 is the unit matrix in Σ, it is now sufficient to deal with ϱ-matrices. This has to be expected since under space inversion the two representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are interchanged. From the Pauli algebra we have:

$$\varrho_3\, 1\, \varrho_3 = 1 , \qquad \varrho_3\, \varrho_3\, \varrho_3 = \varrho_3 , \tag{3.1}$$

$$\varrho_3\, \varrho_1\, \varrho_3 = -\varrho_1 , \quad \varrho_3\, \varrho_1\, \varrho_3\, \varrho_3 = -\varrho_1\, \varrho_3 . \tag{3.2}$$

Hence the elements of the Dirac algebra may be classified as even or odd under the operation of $\gamma_0 \otimes \gamma_0$:

$$\Gamma_e = \{1, \gamma_0, \gamma_i \gamma_5, \sigma_{ik}\} , \qquad \gamma_0\, \Gamma_e\, \gamma_0 = \Gamma_e , \tag{3.3}$$

$$\Gamma_o = \{\gamma_5, \gamma_0 \gamma_5, \gamma_i\, \sigma_{0i}\} , \qquad \gamma_0\, \Gamma_o\, \gamma_0 = -\Gamma_o . \tag{3.4}$$

In the space dependent part only the angles are changed. The parities of the scalar and vector spherical harmonics are well-known:

$$Y_{jj_3}(-\Omega) = (-1)^j\, Y_{jj_3}(\Omega) , \tag{3.5}$$

$$Y_{(jlj_3)i}(-\Omega) = (-1)^l\, Y_{(jej_3)i}(\Omega) . \tag{3.6}$$

From (3.3) – (3.6) we can easily deduce the parity of the sixteen amplitudes. They decouple into two distinct sets of eight amplitudes in each set. For the parity $\eta_P = (-1)^j$ (natural parity) of the bound state $|a\rangle$ we have the amplitudes:

$$s,\ v^{(4)},\ a^{(0)},\ t^{(0)},\ v^{(+)}, v^{(-)},\ u^{(+)},\ u^{(-)}, \eta_P = (-1)^j \tag{3.7}$$

and for $\eta_P = (-1)^{j+1}$ (unnatural parity) we have

$$p, a^{(4)},\ v^{(0)}, u^{(0)},\ a^{(+)}, a^{(-)},\ t^{(+)},\ t^{(-)}, \quad \eta_P = (-1)^{j+1} . \tag{3.8}$$

So the set of sixteen equations decouples into two sets each consisting of eight equations. The solutions of (1.24) are now easily constructed.

4. CP-Invariance

As the next step we consider now CP-invariance, i. e. we are looking for solutions of (1.28). In this case the algebraic part of the problem is not simply discussed in terms of Σ- or P-algebra alone. So we have to deal with the complete Γ-algebra. Again it can be divided into even or odd elements:

$$\Gamma_e' = \{1, \gamma_i, \gamma_i \gamma_5, \sigma_{oi}\}, \quad C\gamma^0 \Gamma_e'^{\mathrm{T}} \gamma^0 C^{-1} = \Gamma_e', \tag{4.1}$$

$$\Gamma_o' = \{\gamma_5, \gamma_o, \gamma_o \gamma_5, \sigma_{ik}\}, \quad C\gamma^0 \Gamma_o'^{\mathrm{T}} \gamma^0 C^{-1} = -\Gamma_o'. \tag{4.2}$$

As concerning the space dependent part, we have to classify the amplitudes as being even or odd under sign change of the relative time. This is done by a subscript "e" or "o". Then we get from (4.1), (4.2) the amplitudes corresponding to a definite CP-eigenvalue ± 1 of the bound state, taking into account the decomposition (3.7), (3.8) because of the parity selection rule.

For $\eta_P = (-1)^j$ and $\eta_P \eta_C = +1$ we have the amplitudes

$$s_e,\ v_o^{(4)},\ a_e^{(0)},\ t_o^{(0)},\ v_e^{(+)},\ v_e^{(-)},\ u_e^{(+)},\ u_e^{(-)} \tag{4.3}$$

and the same amplitudes for $\eta_P = (-1)^j$, $\eta_P \eta_C = -1$ with "e" and "o" interchanged.

For $\eta_P = (-1)^{j+1}$ and $\eta_P \eta_C = -1$ we have the amplitudes:

$$p_e,\ a_e^{(4)},\ v_o^{(0)},\ u_o^{(0)},\ a_o^{(+)},\ a_o^{(-)},\ t_e^{(+)},\ t_e^{(-)} \tag{4.4}$$

and for $\eta_P = (-1)^{j+1}$, $\eta_P \eta_C = -1$ the same with again "e" and "o" interchanged.

Hence CP-invariance does not decouple the set of equations into smaller ones, but allows the restriction of the range of the variable z_0 to positive values.

5. Gourdin Expansion

For the efficient numerical solution of the BS-equation, one has to perform the Wick rotation and go over to Euclidean coordinates. Whether this rotation is allowed, is an entirely analytical problem of the behaviour of the amplitudes at infinity etc. We will not discuss it here. The angular momentum decomposition and the considerations on the discrete symmetries are not affected by this transformation, as is assured by the Hall-Wightman theorem [11]. A more pedestrian way of seeing this is to note that we have never used the reality of the time coordinate z_0.

After Wick rotation, the amplitudes can be expanded in terms of four-dimensional scalar and vector spherical harmonics, defined by [6]:

$$Y_{njj_3}(\Omega) = G_n^{(j)}(\Theta)\, Y_{jj_3}(\vartheta, \varphi), \tag{5.1}$$

$$Y_{(njlj_3)i}(\Omega) = G_n^{(l)}(\Theta)\, Y_{(jlj_3)i}(\vartheta, \varphi) \tag{5.2}$$

where $G_n^{(j)}(\Theta)$ are essentially Gegenbauer polynomials. The angle Θ is defined by $r = R \sin \Theta$, $z_4 = R \cos \Theta$ with z_4 being the rotated time component. The expansion of the amplitudes now reads instead of (2.24):

$$S(z) = \sum_{N=j}^{\infty} s_{N-j}(R)\, Y_{Njj_3}(\Omega) \tag{5.3}$$

and instead of (2.25)

$$\begin{aligned} V_i(z) = \sum_{N=j}^{\infty} \{ & v_{N-j}^{(+)}(R)\, Y_{(N+1j\,j+1j_3)i}(\Omega) \\ & + v_{N-j}^{(0)}(R)\, Y_{(Njjj_3)i}(\Omega) \\ & + v_{N-j}^{(-)}(R)\, Y_{(N-1j\,j-1j_3)i}(\Omega) \}. \end{aligned} \tag{5.4}$$

This is called the Gourdin expansion [12]. If the mass of the bound state is unequal zero, we have no invariance of the equation under four-dimensional rotations. Hence the equations will not decouple with respect to $n := N - j$. The considerations on angular momentum and parity are not altered by the Gourdin expansion. But CP-invariance requires the reversal of the relative time, so we have to discuss this in terms of four-dimensional spherical coordinates. The symmetry properties of the spherical harmonics (5.1), (5.2) with respect to time reversal $(\Theta \to \pi - \Theta)$ are:

$$Y_{Njj_3}(\pi - \Theta, \vartheta, \varphi) = (-1)^{N-j}\, Y_{Njj_3}(\Theta, \vartheta, \varphi), \tag{5.5}$$

$$Y_{(Njlj_3)i}(\pi - \Theta, \vartheta, \varphi) = (-1)^{N-l}\, Y_{(Njlj_3)i}(\Theta, \vartheta, \varphi). \tag{5.6}$$

So the even functions for $z_4 \to -z_4$ have even values of $n = N - j$ in the expansion (5.3), (5.4), and the odd functions odd values. So to (4.3) corresponds the coupling scheme

$$s_{2n},\ v_{2n+1}^{(4)},\ a_{2n}^{(0)},\ t_{2n}^{(+)},\ v_{2n}^{(-)},\ v_{2n}^{(-)},\ u_{2n}^{(+)},\ u_{2n}^{(-)} \tag{5.7}$$

and from (4.4) we have the coupling scheme

$$p_{2n},\ a_{2n}^{(4)},\ v_{2n+1}^{(0)},\ u_{2n+1}^{(0)},\ a_{2n+1}^{(+)},\ a_{2n+1}^{(-)},\ t_{2n+1}^{(+)},\ t_{2n}^{(-)} \tag{5.8}$$

Hence after Gourdin expansion the equation decouples into four distinct sets.

6. Nonrelativistic Quark Model

For a given value j of the angular momentum, we have four distinct solutions corresponding to $\eta_{\mathrm{P}} = \pm(-1)^j$ and $\eta_{\mathrm{P}}\,\eta_{\mathrm{C}} = \pm 1$. We will now discuss the relation of these different states to those obtained from a nonrelativistic quark model.

Of basic importance for this is the equivalence of the CP-transformation with the spin exchange of the particle-antiparticle system [13]. States which are odd under spin exchange are singlet states, and the even ones are triplet states. Hence we have

$$\eta_{\mathrm{P}}\,\eta_{\mathrm{C}} = +1: \text{ triplet states },$$

$$\eta_{\mathrm{P}}\,\eta_{\mathrm{C}} = -1: \text{ singlet states}.$$

Furthermore the parity is given by $-(-1)^l$ nonrelativistically where l is the orbital angular momentum. The additional minus sign stems from the fact that the two particles have different intrinsic parity.

So we have for the unnatural parity $\eta_{\mathrm{P}} = -(-1)^j$ a singlet state with $\eta_{\mathrm{P}}\,\eta_{\mathrm{C}} = -1$, $j = l$ and a triplet state with $\eta_{\mathrm{P}}\,\eta_{\mathrm{C}} = +1$, $j = l$.

For the natural parity $\eta_{\mathrm{P}} = (-1)^j$ the situation is more complicated. It is impossible nonrelativistically to build a singlet state with parity $(-1)^l$. In the quark model these states are called exotics of the second kind [14], and the corresponding particles have not been found so far. In the context of the relativistic BS-equation these states cannot be abandoned by group-theoretical reasons. But it seems possible to exclude them by the normalization condition because they have negative norm at least for zero bound state mass [15]. We will neglect these states in the following. So we have only triplet states with $j = l \pm 1$ which are degenerate.

The possible states are summarized in the following table which is similar to a table in [5]. It should be noted that for $j = 0$ some states are missing because in this case all vector amplitudes with "o" or "–" subscripts vanish. This is easily seen from (2.14).

1 R. F. Keam, Progr. Theor.. Phys. **50**, 957 [1973].
2 M. Böhm, H. Joos, and M. Krammer, Nucl. Phys. **B 51**, 397 [1973].
3 W. Heisenberg, An Introduction to the Unified Theory of Elementary Particles, Wiley, London 1967.
4 R. F. Keam, J. Math. Phys. **9**, 1462 [1968].
5 M. K. Sundaresan and P. J. S. Watson, Ann. Phys. **59**, 375 [1970].
6 K. Ladányi, Ann. Phys. **77**, 471 [1973].
7 R. F. Keam, J. Math. Phys. **10**, 594 [1969].
8 W. Bauhoff and K. Scheerer, Z. Naturforsch. **27 a**, 1539 [1972].
9 A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press, Princeton 1957.
10 H. P. Dürr and F. Wagner, Nuovo Cim. **53 A**, 255 [1968].
11 D. Hall and A. S. Wightman, Dan. Mat. Fys. Medd. **31**, No. 5 [1957].
12 M. Gourdin, Nuovo Cim. **7**, 338 [1958].
13 J. M. Jauch and F. Rohrlich, Theory of Photons and Electrons, Addison-Wesley Publ., Cambridge 1955.
14 H. J. Lipkin, Phys. Rep. **8**, 175 [1973].
15 M. Krammer, DESY T-73/1 [1973].
16 Particle Data Group, Rev. Mod. Phys. **45** [1973].